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Random field Ising model on the Bethe lattice

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Abstract. Low-temperature series expansions have been derived for the random field Ising model with a δ-function distribution on a Bethe lattice by two independent methods: (a) the finite-cluster method which uses graph embeddings and appropriate weighting functions; (b) the use of a recursion relation specific to the Bethe lattice. Numerical values have been evaluated when the coordination number $q=3$, 4 and the coefficients analysed to assess critical behaviour. For small fields, and temperatures near to T_{co}, the critical exponent of the magnetisation seems to retain its mean-field value. But there is clear evidence of a change in critical behaviour at some point on the critical curve. It is argued that when $q>3$ a tricritical point is indicated as found by Aharony in his mean-field solution.

1. Introduction

In a previous paper (Domb and Entin-Wohlman 1984b) we discussed the general derivation of low-temperature or excitation series for the random field Ising model. We showed that the problem is complicated by the existence of a hierarchy of first-order transitions at $T=0$ as the magnetic field is increased from $H=0$. The location and magnitude of these transitions is governed by the structure of clusters in a site percolation process with $p=\frac{1}{2}$. For standard lattices the transitions start at $H=0$, and there is, therefore, no region of non-zero field for which the lowest energy state is one of complete ferromagnetic order.

The only lattice which contains such a region is the Bethe lattice in which there are no surface effects. It seems that this should, therefore, be a suitable lattice for which to derive a low-temperature series.

The finite-cluster method for deriving such series which we discussed in our previous paper can of course be used for the Bethe lattice. But we have found that an alternative approach can be developed specific to this lattice which is rather analogous to the original closed form method used by Bethe (1935). The same series expansions can be derived from the closed form expressions, and this serves as a useful check. The expansions can be extended without too much difficulty using this approach and certain groups of terms can be partially summed. We shall find this property useful in trying to assess the asymptotic behaviour of the series expansions.

2. The model

We first introduce the notation which we shall use to identify the sites in the Bethe lattice (a typical such lattice with coordination number $q=3$ is depicted in figure 1).

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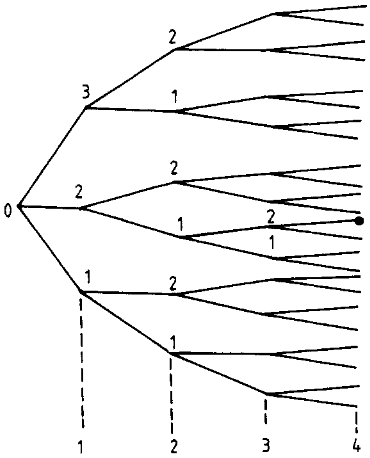

Figure 1. Schematic diagram of Bethe lattice illustrating notation for successive generations. The full circle indicates a point on the fourth generation.

The suffix k is used as a generation index, and ν_k represents a lattice site of the kth generation. Each ν_k is specified precisely by its ancestry $(i_i, i_2, i_3, \ldots, i_k)$ where i_r represents a particular branch of the rth generation. i_1 goes from 1 to q and the remaining $i_r (r>1)$ go from 1 to $(q-1)$. A typical ν_4 site marked on the figure is represented by $(2, 1, 2, 2)$.

The Hamiltonian is given by

$$\mathscr{H} = \sum_k \left(\sum_{\nu_k} (-J\sigma_{\nu_k}\sigma_{\nu_{k+1}} - H_{\nu_{k+1}}\sigma_{\nu_{k+1}}) \right) \tag{1}$$

where the index k runs through $k = 0, 1, 2, \ldots$, and $k = 0$ represents the central site. The σ_i take on values ± 1, and the $\nu_k \nu_{k+1}$ sum is taken over all connected pairs in the kth and $(k+1)$th generations; $H_{\nu_{k+1}}$ is the random field at site ν_{k+1}.

For the treatment which follows we have derived benefit from Stein (1983). We wish to derive an expression for the partition function Z from the trace of $\exp(-\beta\mathscr{H})$, and we start by taking the trace over the spins of the $(k+1)$th generation.

This yields

$$\prod_{\nu_{k+1}} 2\cosh(\beta J\sigma_{\nu_k} + \beta H_{\nu_{k+1}}) \tag{2}$$

which we rewrite in the form

$$\begin{aligned} &\prod_{\nu_{k+1}} \exp \ln[2\cosh(\beta J\sigma_{\nu_k} + \beta H_{\nu_{k+1}})] \\ &\qquad = \prod_{\nu_{k+1}} \exp[\tfrac{1}{2}(1+\sigma_{\nu_k}) \ln 2\cosh(\beta J + \beta H_{\nu_{k+1}}) \\ &\qquad\qquad + \tfrac{1}{2}(1-\sigma_{\nu_k}) \ln 2\cosh(\beta J - \beta H_{\nu_{k+1}})] \\ &\qquad = \prod_{\nu_{k+1}} \exp\left(\tfrac{1}{2} \ln 4\cosh(\beta J + \beta H_{\nu_{k+1}}) \cosh(\beta J - \beta H_{\nu_{k+1}}) \right. \\ &\qquad\qquad \left. + \tfrac{1}{2}\sigma_{\nu_k} \ln \frac{\cosh(\beta J + \beta H_{\nu_{k+1}})}{\cosh(\beta J - \beta H_{\nu_{k+1}})} \right). \end{aligned} \tag{3}$$

It follows that taking the trace over the $(k+1)$th generation gives a factor

$$\phi_k = \exp\left(\sum_{\nu_{k+1}} \tfrac{1}{2} \ln 4 \cosh(\beta J + \beta H_{\nu_{k+1}}) \ln \cosh(\beta J - \beta H_{\nu_{k+1}})\right) \tag{4}$$

and modifies the field acting on the spins ν_k replacing βH_{ν_k} by

$$L_{\nu_k} = \beta H_{\nu_k} + \tfrac{1}{2} \sum_{i_{k+1}} \ln \frac{\cosh(\beta J + \beta H_{\nu_{k+1}})}{\cosh(\beta J - \beta H_{\nu_{k+1}})}. \tag{5}$$

But if the trace has already been taken over generations $(k+2)$ to ∞, then all the H_{ν_i} from $(k+1)$ to infinity have been replaced by L_{ν_i}. Hence (4) must be replaced by

$$\phi_k = \exp\left(\sum_{\nu_{k+1}} \tfrac{1}{2} \ln 4 \cosh(\beta J + \beta L_{\nu_{k+1}}) \ln \cosh(\beta J - \beta L_{\nu_{k+1}})\right) \tag{6}$$

and (5) by

$$L_{\nu_k} = \beta H_{\nu_k} + \tfrac{1}{2} \sum_{\nu_{k+1}} \ln \frac{\cosh(\beta J + \beta L_{\nu_{k+1}})}{\cosh(\beta J - \beta L_{\nu_{k+1}})}. \tag{7}$$

Repeating the above argument for the infinite lattice and tracing over all generations except the zeroth, we find that

$$Z(\sigma_{\nu_0}) = \exp(L_{\nu_0}\sigma_{\nu_0}) \exp\left(\sum_{k=1}^{\infty} \sum_{\nu_k} \tfrac{1}{2} \ln[4 \cosh(\beta J + L_{\nu_k}) \cosh(\beta J - L_{\nu_k})]\right). \tag{8}$$

Note that the sums in (6), (7) and (8) contain $(q-1)$ terms for any ν_k other than ν_0, but for ν_0 they contain q terms. From (8) we can readily derive the magnetisation of the central spin

$$M = \tanh L_{\nu_0}. \tag{9}$$

We shall find it convenient to write equation (7) in the form

$$L_{\nu_k} = \beta H_{\nu_k} + A_{\nu_k} \tag{10}$$

where

$$A_{\nu_k} = \tfrac{1}{2} \sum_{\nu_{k+1}} \ln \frac{1 + w\tau_{\nu_{k+1}}}{1 - w\tau_{\nu_{k+1}}} \tag{11}$$

and we have written

$$w = \tanh \beta J, \qquad \tau_{\nu_k} = \tanh L_{\nu_k}. \tag{12}$$

We shall also write

$$\theta_{\nu_k} = \exp(-2\beta L_{\nu_k}) = (1 - \tau_{\nu_k})/(1 + \tau_{\nu k}) \tag{13}$$

θ_{ν_0} is given from equation (7) by

$$\theta_{\nu_0} = \exp(-2\beta H_{\nu_0}) \prod_{i_1=1}^{q} \frac{z + \theta_{\nu_1}}{1 + z\theta_{\nu_1}} \qquad (z = \exp(-2\beta J)). \tag{14}$$

The variables θ_{ν_k} associated with all lattice sites except the central site ν_0 satisfy

$$\theta_{\nu_k} = \exp(-2\beta H_{\nu_k}) \prod_{i_{k+1}=1}^{q-1} \frac{z + \theta_{\nu_{k+1}}}{1 + z\theta_{\nu_{k+1}}}. \tag{15}$$

In terms of the variables θ the magnetisation (equation (9)) is given by

$$M=(1-\theta_{\nu_0})/(1+\theta_{\nu_0}). \tag{16}$$

3. Magnetisation at low temperatures

So far our treatment has been exact, and could be used for a model with different magnetic fields at each site. We now go over to a stochastic model in which the magnetic field follows a probabilty distribution which is identical for all sites.

We first note from (11) that the average of A_{ν_k} depends only upon the odd moments of τ_{ν_k} (i.e. $\langle\tau_{\nu_k}^{2l+1}\rangle$). Thus, for a random field distribution $P(H)$ which is symmetric about $H=0$ there is always a solution $\langle\tau_{\nu_k}^{2l+1}\rangle=0$ which yields zero magnetisation. Our aim is to search for another solution which will yield a finite value for the magnetisation.

Denote the random-field moments by m_l

$$m_l=\langle\exp(-2l\beta H_{\nu_k})\rangle, \tag{17}$$

and the θ-moments (which are identical for all lattice sites with $k>0$) by

$$\theta_l=\langle\theta_{\nu_k}^l\rangle \qquad (k>0). \tag{18}$$

We can average equation (15) to obtain

$$\begin{aligned}
\theta_1&=m_1[z+(1-z^2)\theta_1-z(1-z^2)\theta_2+z^2(1-z^2)\theta_3\ldots]^{q-1}\\
\theta_2&=m_2[z^2+2z(1-z^2)\theta_1+(1-z^2)(1-3z^2)\theta_2\ldots]^{q-1}\\
\theta_3&=m_3[z^3+3z^2(1+z^2)\theta_1+3z(1-z^2)(1-2z^2)\theta_2\ldots]^{q-1}\\
&\ldots\ldots
\end{aligned} \tag{19}$$

Following the same procedure for θ_{ν_0} for the central site, we find from (14)

$$\begin{aligned}
\langle\theta_{\nu_0}\rangle&=m_1[z+(1-z^2)\theta_1-z(1-z^2)\theta_2+z^2(1-z^2)\theta_3\ldots]^{q}\\
\langle\theta_{\nu_0}^2\rangle&=m_2[z^2+2z(1-z^2)\theta_1+(1-z^2)(1-3z^2)\theta_2+\ldots]^{q}\\
\langle\theta_{\nu_0}^3\rangle&=m_3[z^3+3z^2(1-z^2)\theta_1+3z(1-z^2)(1-2z^2)\theta_2+\ldots]^{q}\\
&\ldots\ldots
\end{aligned} \tag{20}$$

To find the magnetisation (16) equations (19) must be solved for θ_1, θ_2, $\theta_3\ldots$ and the results inserted in equations (20).

From the structure of equations (16), (19) and (20), it will be seen that θ_1, θ_2, .. (which are all positive) must be small quantities in order to obtain a finite positive value for the magnetisation ($\leqslant 1$). In seeking an iterative solution for θ_l we note that for the early terms in the brackets on the right-hand side of (19) to be of comparable size as $T\to 0$ we require

$$\theta_1\sim z, \qquad \theta_2\sim z^2, \qquad \theta_3\sim z^3, \qquad \ldots\theta_l\sim z^l. \tag{21}$$

When this condition is satisfied, we find from (19) that

$$\theta_l\sim m_l z^{l(q-1)}, \tag{22}$$

and hence from (21)

$$m_l z^{l(q-2)} \sim 1. \tag{23}$$

For a δ-function distribution

$$P(H) = \tfrac{1}{2}[\delta(H - H_0) + \delta(H + H_0)], \tag{24}$$

criterion (23) becomes

$$\tfrac{1}{2}(\exp\{-2\beta l[H_0 - (q-2)J]\} + \exp\{-2\beta l[H_0 + (q-2)J]\}) \sim 1. \tag{25}$$

Hence as $T \to 0$ and $\beta \to \infty$ we find the condition

$$H_0 \leqslant (q-2)J. \tag{26}$$

For a Gaussian distribution

$$P(H) = (1/\sigma(2\pi)^{1/2}) \exp(-H^2/2\sigma^2) \tag{27}$$

we find that

$$m_l \sim \exp(2\beta^2 l^2 \sigma^2), \tag{28}$$

and consequently condition (23) can never be satisfied, however small σ. This corresponds physically to the result we have found previously that however small σ the lowest energy state will contain overturned spins and will not correspond to complete ferromagnetic order.

When the distribution is bounded condition (23) can always be satisfied if the distribution bounds $\pm H_0$ satisfy (26). This is because in the small T limit we need

$$\begin{aligned}\exp[-2\beta lJ(q-2)] &\int_{-H_0}^{0} [\exp(-2\beta lH)]P(H)\,\mathrm{d}H \\ &= \exp\{2\beta l[H_0 - (q-2)J]\} \int_0^{H_0} \{\exp[2\beta l(H - h_0)]\} P(H)\,\mathrm{d}H\end{aligned} \tag{29}$$

to be finite; this is satisfied if $H_0 \leqslant (q-2)J$.

We have solved equations (19) and (20) by iteration and obtained the first terms of the low-temperature series for the magnetisation for $q = 3, 4$. The results are listed in the appendix. The method of finite clusters which is applicable to all lattices has already been described, (Domb and Entin-Wohlman 1984b) and yields the free energy series at the same time as the magnetisation series. The iterative method for the Bethe lattice, yields the magnetisation series in the first instance, but the free energy series could be obtained from (8). We were gratified to find that both methods yielded the same results for the magnetisation series. The iterative method could be pursued much further if necessary.

4. Analysis of the magnetisation series

We have seen that low-temperature series based on a completely ordered ferromagnetic ground state can be derived only for a distribution of random field which is bounded. We therefore confine ourselves to a δ-function distribution (24) with H_0 satisfying (26).

For the standard Ising model on a Bethe lattice mean-field exponents are obtained, and the spontaneous magnetisation tends to zero as $(T_{co}-T)^{1/2}$, with

$$\tanh \beta_{co}J = (q-1)^{-1}. \tag{30}$$

There is a good deal of theoretical support for the conclusion that in a random field on the Bethe lattice the exponents retain their mean field values (see e.g. Aharony *et al* 1976). We have therefore inverted the magnetisation series to obtain M^{-2} in the form

$$M^{-2} = 1 + a_1 z + a_2 z^2 + \ldots \tag{31}$$

where the coefficients a_l are functions of the field moments m_l. Fortunately the coefficients are all positive in sign, and hence the ratio

$$\mu_n = a_n / a_{n-1} \to \mu + \mathrm{o}(n) \tag{32}$$

will lead us to the physical singularity.

We are here concerned with a double series in some respects analogous to that for an Ising antiferromagnet in an external field (see e.g. Domb 1974). A detailed analysis of critical behaviour requires a good deal of sophistication, and we defer such a treatment to a separate publication. However, it is possible to draw some conclusions from a simple ratio analysis.

For the standard Ising model all the ratios μ_n in (32) are equal, (e.g. $\mu_n = 3$ for $q = 3$) and $\mu_n z = 1$ locates the transition temperature. In the random field case, if we put $\mu_n z = 1$ we shall obtain a family of curves in the $H_0 - T$ plane, and if the curves for successive n fall closely on one another we can conclude that the critical exponent retains its mean-field value.

We have plotted these curves for $q = 4$ (figure 2) and $q = 3$ (figure 3), and have found that the curve resulting from $\mu_n z = 1$ lies below that resulting from $\mu_{n-1} z = 1$. At low fields and temperatures near T_{c0} the curves fall very nearly on one another. However for a particular field H_{0t} the curves seem to separate out, and we interpret this as corresponding to a change of power of n in the coefficients of series (31).

Fortunately we can obtain some guidance on the nature of the change at H_{0t} by looking in detail at the behaviour near $T = 0$. At $T = 0$ we know that the system has a first-order transition (Domb and Entin-Wohlman 1984a). If we look at the series for the magnetisation given in the appendix ((A2) and (A4)) we find that at low temperatures the positive powers of m_1 are dominant. We shall find that we can sum them readily for $q = 3$; in fact we can sum all the terms involving only powers of m_1. To do this we put $m_2 = m_3 = \ldots = 0$ in equation (19); the only non-zero θ is θ_1 which is determined by the equation

$$\theta_1 = m_1[z + (1-z^2)\theta_1]^2 \qquad (\theta_1/m_1)^{1/2} = z + (1-z^2)\theta_1. \tag{33}$$

Ignoring z^2 in comparison with 1 we obtain the solution

$$\theta_1^{1/2} = \tfrac{1}{2} m_1^{-1/2}[1 - (1 - 4m_1 z)^{1/2}]. \tag{34}$$

The magnetisation is then derived from (20) and (16)

$$\langle \theta_{\nu_0} \rangle = m_1[z + (1-z)^2\theta_1]^3 = m_1^{-1/2}\theta_1^{3/2}, \tag{35}$$

and

$$M^{-2} = 1 + 4\langle \theta_{\nu_0} \rangle = 1 + \tfrac{1}{2} m_1^{-2}[1 - (1 - 4m_1 z)^{1/2}]^3. \tag{36}$$

The analytic behaviour of (36) is clear; there is a singularity at points given by

$$4m_1 z = 1, \tag{37}$$

beyond which M^{-2} has no real value and at the singularity M^{-2} takes on the value $1+8z^2$. We shall discuss the nature of the singularity in the next section. If (36) is expanded as a power series, the dominant asymptotic contribution comes from the term $-(1-4m_1z)^{1/2}$, and we obtain an asymptotic approximation to the coefficient of z^n of the form

$$4^n m_1^{n-2} n^{-3/2}. \tag{38}$$

It is the $n^{-3/2}$ factor which causes the fanning out in figure 3.

The same analysis can be carried whilst retaining the $(1-z^2)$ factor in (33), and we find that the singularity then corresponds to

$$4m_1 z(1-z^2) = 1, \tag{39}$$

and M^{-2} takes on the value

$$1+8z^2/(1-z^2). \tag{40}$$

Curve (37) has been drawn in figure 3 and will be seen to be quite close to the computed curves.

For $q=4$ we can adopt the same procedure, but the equation to determine θ_1 replacing (33) is now a cubic,

$$(\theta_1/m_1)^{1/3} = z+(1-z^2)\theta_1. \tag{41}$$

An explicit solution could be found by Cardan's method but is rather complicated, so we have not pursued the matter. It is easy to find the critical curve analogous to (39) from the condition that the line $y = m_1^{-1/3}x$ should be a tangent to the curve

$$y = z+(1-z^2)x^3. \tag{42}$$

We find for this curve the relation

$$\tfrac{27}{4}m_1 z^2(1-z^2) = 1, \tag{43}$$

and it is plotted in figure 2. The corresponding value of M^{-2} is

$$M^{-2} = 1+3z^2/(1-z^2).$$

5. Nature of the transition

We have discussed the general nature of the transition as a function of magnetic field at $T=0$ in previous papers (Domb and Entin-Wohlman 1984a, b) but we shall now concentrate specifically on the Bethe lattice. The coupling energy lost in overturning a cluster of n connected sites on this lattice is

$$[(q-2)n+2]2J \tag{44}$$

whereas the gain in magnetic energy is $2nH_{0n}$. Hence the critical magnetic field for the overturn of such a cluster is

$$H_{0n} = (q-2)J+(2/n)J. \tag{45}$$

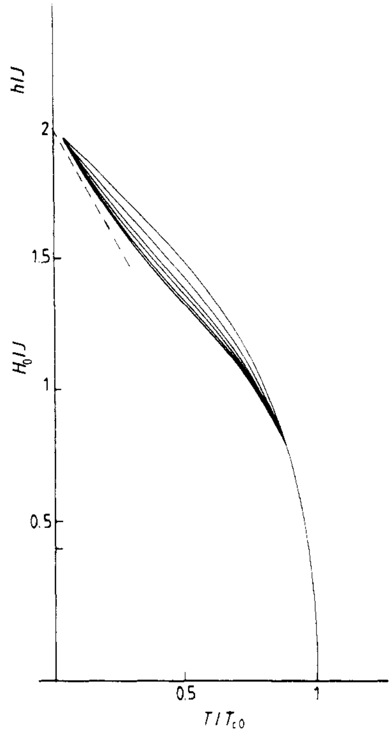

Figure 2. $q = 4$. Curves $\mu_n z = 1$ for successive $n(\mu_n = a_n/a_{n-1}$ in the series expansion for M^{-2}. The broken curve, $\frac{27}{4}mz^2 = 1$, corresponds to the summation of all the leading powers of m_1.

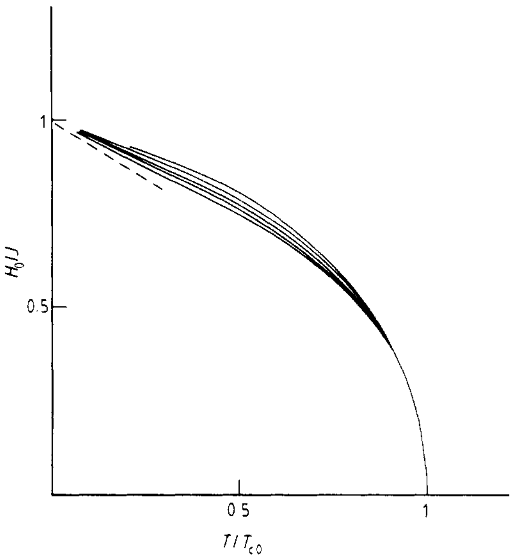

Figure 3. $q = 3$. Curves $\mu_n z = 1$ as in figure 2. The broken curve is now $4m_1 z = 1$.

We thus have an infinite set of first-order phase transitions at points given by $n = 1, 2, 3, \ldots$, at fields varying from qJ to $(q-2)J$.

If there is an infinite cluster in the site percolation process for $p = \frac{1}{2}$, we can say that there is a first-order transition at a field equal to $(q-2)J$ corresponding to $n = \infty$. There will be an infinite cluster when $q > 3$ and the magnitude of the discontinuity in magnetisation is given by the fraction of sites in the infinite cluster.

We can then say from our previous discussion and equation (43) that this first-order transition continues when $T > 0$. Since we have provided good evidence when $q = 3, 4$ that for small fields near to T_{c0} there is a range of H_0 for which the system undergoes a second-order transition, it is reasonable to conclude that at some point (H_{0t}, T_{ct}) on the critical curve there is a tricritical point marking the change from a first-order to a second-order transition.

Aharony (1978) obtained the same result in a mean field approximation, but our critical value of field H_0 at $T = 0$, $(q-2)J$, differs from his which is $\frac{1}{2}qJ$. The nature of this discrepancy needs further investigation.

When $q = 3$ there is no infinite cluster at $p = \frac{1}{2}$, and the transition at $(q-2)J$ is to a limit point of first-order transitions. However there is an incipient infinite cluster since $p_c = \frac{1}{2}$. Because of the analogy between equations (39), (40) and (43) the general pattern along the transition curve seems to be the same as for $q = 4$, but the detailed behaviour of the singularities for $T > 0$ and $H_0 > (q-2)J$ need further investigation.

6. Conclusions

We have provided substantial evidence to support the conclusion that the behaviour of the random-field Ising model on the Bethe lattice for $q > 3$ is the same as that obtained by Aharony (1978) in the mean-field approximation. But it is clear that our treatment is specific to the Bethe lattice since for all standard lattices overturned clusters occur down to $H_0 = 0$.

To explore the behaviour of standard lattices at low temperatures as well as that of the Bethe lattice for $H_0 > (q-2)J$ it is necessary to develop low-temperature expansions for the case in whch the ground state is not one of complete ferromagnetic order. We are hoping to deal with this in the near future.

Acknowledgment

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Appendix. Low-temperature series for $q = 3, 4$

The magnetisation series were obtained by two methods. The first is the iterative solution of equations (19) and (20). The second is based upon the excitation expansion (Domb 1974, Domb and Entin-Wohlman 1984b). In the process of the second method we also obtained the low-temperature series for the free energy. For the sake of completeness, we list these as well.

(a) Four nearest neighbours ($q = 4$)

$$\begin{aligned}\langle \ln Z \rangle = 2\beta J &+ z^4 m_1 + z^6 2m_1^2 + z^8(6m_1^3 - 2m_1^2 - \tfrac{1}{2}m_2) + z^{10}(22m_1^4 - 12m_1^3 - 4m_1m_2) \\ &+ z^{12}(91m_1^5 - 66m_1^4 - 24m_1^2m_2 - m_2^2 + 6m_1^3 + 4m_1m_2 + \tfrac{1}{2}m_3) \\ &+ z^{14}(408m_1^6 - 364m_1^5 - 136m_1^3m_2 - 12m_2^2m_1 + 66m_1^4 \\ &+ 48m_1^2m_2 + 4m_2^2 + 4m_1m_3) \\ &+ z^{16}(1938m_1^7 - 2040m_1^6 - 768m_1^4m_2 - 96m_2^2m_1^2 - 3m_2^3 + 546m_1^5 + 408m_1^3m_2 \\ &+ 66m_2^2m_1 + 30m_1^2m_3 + 4m_2m_3 + 22m_1^4 - 24m_1^2m_2 - 3m_2^2 - 4m_1m_3 - \tfrac{1}{4}m_4).\end{aligned} \tag{A1}$$

The random-field moments m_l are given by equation (17). The magnetisation per site is

$$\begin{aligned}M = 1 - 2[&z^4m_1 + z^6 4m_1^2 + z^8(18m_1^3 - 4m_1^2 - m_2) + z^{10}(88m_1^4 - 36m_1^3 - 12m_1m_2) \\ &- z^{12}(455m_1^5 - 264m_1^4 - 96m_1^2m_2 - 4m_2^2 + 18m_1^3 + 12m_1m_2 + m_3) \\ &+ z^{14}(2448m_1^6 - 1820m_1^5 - 680m_1^3m_2 - 60m_2^2m_1 + 264m_1^4 + 192m_1^2m_2 \\ &+ 16m_2^2 + 16m_1m_3) + z^{16}(13\,566m_1^7 - 12\,240m_1^6 - 4608m_1^4m_2 - 576m_2^2m_1^2 \\ &- 18m_2^3 + 2730m_1^5 + 2040m_1^3m_2 + 330m_2^2m_1 + 150m_1^2m_3 + 20m_2m_3 - 88m_1^4 \\ &- 96m_1^2m_2 - 12m_2^2 - 12m_1m_3m_4)].\end{aligned} \tag{A2}$$

(b) Three nearest neighbours ($q = 3$)

$$\langle \ln z\rangle = \tfrac{3}{2}\beta J + z^3 m_1 + z^4 \tfrac{3}{2} m_1^2 + z^5 3 m_1^3 + z^6(7m_1^4 - \tfrac{3}{2}m_1^2 - \tfrac{1}{2}m_2)$$
$$+ z^7(18m_1^5 - 6m_1^3 - 3m_1 m_2) + z^8(\tfrac{99}{2}m_1^6 - 21 m_1^4 - 12 m_2 m_1^2 - \tfrac{3}{4}m_2^2)$$
$$+ z^9(143 m_1^7 - 72 m_1^5 - 43 m_1^3 m_2 - 6 m_2^2 m_1 + 3m_1^3 + 3 m_1 m_2 + \tfrac{1}{3}m_3) \qquad (A3)$$

$$M = 1 - 2[z^3 m_1 + z^4 3 m_1^2 + z^5 9 m_1^3 + z^6(28 m_1^4 - 3m_1^2 - m_2)$$
$$+ z^7(90 m_1^5 - 18 m_1^3 - 9 m_1 m_2) + z^8(297 m_1^6 - 84 m_1^4 - 48 m_2 m_1^2 - 3m_2^2)$$
$$+ z^9(1001 m_1^7 - 360 m_1^5 - 215 m_1^3 m_2 - 30 m_2^2 m_1 + 9m_1^3 + 9 m_1 m_2 + m_3)$$
$$+ z^{10}(3432 m_1^8 - 1485 m_1^6 - 900 m_1^4 m_2 - 180 m_2^2 m_1^2 - 9 m_2^3 + 84 m_1^4 + 96 m_1^2 m_2$$
$$+ 12 m_1 m_3 + 12 m_2^2) + z^{11}(11\,934 m_1^9 - 6006 m_1^7 - 3654 m_1^5 m_2 - 882 m_1^3 m_2^2$$
$$- 105 m_1 m_2^3 + 540 m_1^5 + 645 m_1^3 m_2 + 75 m_1^2 m_3 + 165 m_1 m_2^2 + 15 m_2 m_3)]. \qquad (A4)$$

In this latter case we have derived two additional terms in M by the iteration method. The series for M^{-2} discussed in the text were obtained by squaring and inverting equations (A2) and (A4).

Note. Soon after this paper was completed Dr A Aharony passed on to us a copy of an interesting IBM preprint by R Bruinsma, dealing with the same topic. Bruinsma deals with the case $q = 3$ and obtains an integral equation from the recursion relation which he solves by iteration. His work is thus complementary to ours. Some of the results he obtains (i.e. the suggested absence of a tricritical point) may be specific to $q = 3$ for which there is no infinite cluster at $p = \frac{1}{2}$.

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